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ON ORTHOGONAL LATIN p-DIMENSIONAL CUBES

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Abstract. We give a construction of p orthogonal Latin p-dimensional cubes (or Latin hypercubes) of order n for every natural number $n \neq 2, 6$ and $p \geqslant 2$. Our result generalizes the well known result about orthogonal Latin squares published in 1960 by R. C. Bose, S. S. Shikhande and E. T. Parker.

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In 1960, R. C. Bose, S. S. Shikhande and E. T. Parker [1] proved that two orthogonal Latin squares of order n exist if and only if $n \neq 2, 6$. (For more information about these topics see [2] and [3].)

A generalization of Latin squares are Latin p-dimensional cubes (sometimes called *Latin hypercubes*). In this paper we generalize the well know result from [1] into p-dimensional space for every natural number p.

Definition. A *Latin p-dimensional cube* of order n is a p-dimensional matrix

$$\mathbf{Q}^{p,n} = |\mathbf{q}(i_1, i_2, \ldots, i_p);\ 1 \leqslant i_1, i_2, \ldots, i_p \leqslant n|,$$

such that every row is a permutation of the set of natural numbers $1, 2, \ldots, n$. By a *row* of $\mathbf{Q}^{p,n}$ we mean an n-tuple of elements $\mathbf{q}(i_1, i_2, \ldots, i_p)$ which have identical coordinates at $p-1$ places.

Definition. A p-tuple of Latin p-dimensional cubes

$$[\mathbf{Q}_k^{p,n} = |\mathbf{q}_k(i_1, i_2, \ldots, i_p);\ 1 \leqslant i_1, i_2, \ldots, i_p \leqslant n|,\ k = 1, 2, \ldots, p]$$

of order n is called *orthogonal*, if whenever $i_1, i_2, \ldots, i_p, i'_1, \ldots, i'_p \in \{1, 2, \ldots, n\}$ are such that

$$\mathbf{q}_k(i_1, i_2, \ldots, i_p) = \mathbf{q}_k(i'_1, i'_2, \ldots, i'_p) \quad \text{for all } \ k = 1, 2, \ldots, p,$$

then we must have $i_k = i'_k$ for all $k = 1, 2, \ldots, p$.

The construction of a p-tuple of orthogonal Latin p-dimensional cubes is contained in the proof of the following theorem.

Theorem. *A p-tuple of orthogonal Latin p-dimensional cubes $\mathbf{Q}_k^{p,n}$ of order n exists for every natural number $n \neq 2, 6$ and every natural number $p \geqslant 2$.*

Proof. Let $\mathbf{R}^n = |\mathbf{r}(i_1, i_2);\ 1 \leqslant i_1, i_2 \leqslant n|$ and $\mathbf{S}^n = |\mathbf{s}(i_1, i_2)|$ be two orthogonal Latin squares of order n. They will have a crucial role in our construction of p orthogonal Latin p-dimensional cubes $\mathbf{Q}_k^{p,n}$, $k = 1, 2, \ldots, p$. The k-th cube arises using the square $\mathbf{R}^n$ $(k-1)$-times and the square $\mathbf{S}^n$ $(p-k)$-times.

We define the k-th Latin p-dimensional cube

$$\mathbf{Q}_k^{p,n} = |\mathbf{q}_k(i_1, i_2, \ldots, i_p)|$$

of order n by the following relation

$$\begin{gathered}\mathbf{q}_k(i_1, \ldots, i_p) \\ = \mathbf{r}(i_1, \mathbf{r}(i_2, \mathbf{r}(i_3, \ldots, \mathbf{r}(i_{k-1}, \mathbf{s}(i_k, \mathbf{s}(i_{k+1}, \ldots, \mathbf{s}(i_{p-2}, \mathbf{s}(i_{p-1}, i_p)) \ldots))) \ldots)))\end{gathered}$$

for every $1 \leqslant i_1, i_2, \ldots, i_p \leqslant n$.

1. Evidently, for every $k = 1, 2, \ldots, p$, the set

$$\{\mathbf{q}_k(i_1, i_2, \ldots, i_{j-1}, i_j, i_{j+1}, \ldots, i_p);\ i_j = 1, 2, \ldots, n\}$$

is equal to the set $\{1, 2, \ldots, n\}$. From this it follows that $\mathbf{Q}_k^{p,n}$ is a Latin p-dimensional cube for all k.

2. Suppose that

$$(\mathrm{E}_k) \qquad \mathbf{q}_k(i_1, i_2, \ldots, i_p) = \mathbf{q}_k(i'_1, i'_2, \ldots, i'_p) \quad \text{for all } \ k = 1, 2, \ldots, p.$$

From (E_1) and (E_2) it follows that

$$\begin{aligned}\mathbf{s}(i_1, \mathbf{s}(i_2, \mathbf{s}(i_3, \ldots, \mathbf{s}(i_{p-1}, i_p) \ldots))) &= \mathbf{s}(i'_1, \mathbf{s}(i'_2, \mathbf{s}(i'_3, \ldots, \mathbf{s}(i'_{p-1}, i'_p) \ldots))),\\ \mathbf{r}(i_1, \mathbf{s}(i_2, \mathbf{s}(i_3, \ldots, \mathbf{s}(i_{p-1}, i_p) \ldots))) &= \mathbf{r}(i'_1, \mathbf{s}(i'_2, \mathbf{s}(i'_3, \ldots, \mathbf{s}(i'_{p-1}, i'_p) \ldots))).\end{aligned}$$

Because $\mathbf{R}^n$ and $\mathbf{S}^n$ are orthogonal Latin squares, we have

$$i_1 = i_1'$$

and

$$\mathbf{s}(i_2, \mathbf{s}(i_3, \ldots, \mathbf{s}(i_{p-1}, i_p) \ldots)) = \mathbf{s}(i_2', \mathbf{s}(i_3', \ldots, \mathbf{s}(i_{p-1}', i_p') \ldots)).$$

Replace i_1' by i_1 in (E_k), $k = 1, 2, \ldots, p$. From (E_2) and (E_3) it follows that

$$\begin{aligned}\mathbf{s}(i_2, \mathbf{s}(i_3, \ldots, \mathbf{s}(i_{p-1}, i_p) \ldots))) &= \mathbf{s}(i_2', \mathbf{s}(i_3', \ldots, \mathbf{s}(i_{p-1}', i_p') \ldots))),\\ \mathbf{r}(i_2, \mathbf{s}(i_3, \ldots, \mathbf{s}(i_{p-1}, i_p) \ldots))) &= \mathbf{r}(i_2', \mathbf{s}(i_3', \ldots, \mathbf{s}(i_{p-1}', i_p') \ldots))),\end{aligned}$$

and so

$$i_2 = i_2'$$

and

$$\mathbf{s}(i_3, \mathbf{s}(i_4, \ldots, \mathbf{s}(i_{p-1}, i_p) \ldots)) = \mathbf{s}(i_3', \mathbf{s}(i_4', \ldots, \mathbf{s}(i_{p-1}', i_p') \ldots)).$$

Continuing in this manner, after $(p-1)$ steps from (E_{p-1}) and (E_p) we get

$$\begin{aligned}\mathbf{s}(i_{p-1}, i_p) &= \mathbf{s}(i_{p-1}', i_p'),\\ \mathbf{r}(i_{p-1}, i_p) &= \mathbf{r}(i_{p-1}', i_p').\end{aligned}$$

From the assumption that $\mathbf{R}^n$ and $\mathbf{S}^n$ are orthogonal we get

$$i_{p-2}' = i_{p-2} \quad \text{and} \quad i_{p-1}' = i_{p-1},$$

which completes the proof of orthogonality. $\square$

Remark 1. Our construction is based on a pair of orthogonal Latin squares and so we give no information about Latin p-dimensional cubes of order 2 and 6.

Remark 2. If n is odd then $\mathbf{R}^n = |\mathbf{r}(i_1, i_2) = (i_1 + i_2) \pmod n;\ 1 \leqslant i_1, i_2 \leqslant n|$ and $\mathbf{S}^n = |\mathbf{s}(i_1, i_2) = (i_1 - i_2) \pmod n|$ are mutually orthogonal Latin squares. Using these two squares the formula for making a magic p-dimensional cube of odd order was derived. (See [4].)

References

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